Tau-functions as highest weight vectors for $W_{1+\infty}$ algebra

1996 J. Phys. A: Math. Gen. 29 5565

(http://iopscience.iop.org/0305-4470/29/17/027)

J. Phys. A: Math. Gen. **29** (1996) 5565–5573. Printed in the UK

Tau-functions as highest weight vectors for $W_{1+\infty}$ algebra

B Bakalov†, E Horozov‡ and M Yakimov§

Department of Mathematics and Informatics, Sofia University, 5 J Bourchier Blvd, Sofia 1126, Bulgaria

Received 11 December 1995

Abstract. For each $\boldsymbol{r} = (r_1, r_2, \ldots, r_N) \in \mathbb{C}^N$ we construct a highest weight module $\mathcal{M}_{\boldsymbol{r}}$ of the Lie algebra $W_{1+\infty}$. The highest weight vectors are specific tau-functions of the Nth Gelfand–Dickey hierarchy. We show that these modules are quasifinite and we give a complete description of the reducible ones together with a formula for the singular vectors.

1. Introduction

The remarkable connection between the infinite-dimensional Lie algebras and the soliton equations was noticed by Sato [24] and further developed by Date *et al* in [7]. In particular it was found that the Kac–Moody algebras and the Virasoro algebra (important for the conformal field theory) play a substantial role in soliton theory (see [2, 26, 8], etc for more details). We would only mention the work of several authors (see [21, 22, 29] and references therein) where it was discovered that the partition function of 2D quantum gravity is a tau-function for the KdV hierarchy and also satisfies the so-called Virasoro constraints. This result can also be interpreted as a construction of a certain highest weight representation of the Virasoro algebra. Later a whole class of representations of the Virasoro algebra in terms of tau-functions was built in [14, 15]. Certain special functions like Airy or Bessel functions and Hermite or Laguerre polynomials play an important role in all above-mentioned results.

The present paper deals with similar questions but for the Lie algebra $W_{1+\infty}$. This algebra is the unique central extension of the Lie algebra of regular differential operators on the circle [17]. In recent works (see e.g. references in [11, 3]) this algebra and its reductions W_N were found to play an important role in quantum field theory. $W_{1+\infty}$ is also the algebra of the additional symmetries of the KP tau-functions [26]. The representation theory of $W_{1+\infty}$ was recently initiated in [19, 11, 3], etc. In particular Kac and Radul isolated a class of $W_{1+\infty}$-modules and classified them. These are graded modules with finite-dimensional level spaces, called in [19] quasifinite.

In contrast to the general theory we are interested in concrete representations connected to classical special functions—this time—Meijer's G-functions (see [6, 23]). Our construction uses a simple but beautiful idea of Kac and Schwarz [21]. We recall it briefly. Each tau-function corresponds to a plane W from the Sato Grassmannian Gr which can be considered as an infinite wedge product $|W\rangle$. Assume that an operator A leaves the plane W invariant. Then under the boson–fermion correspondence σ the image $\tau_W(t) = \sigma(|W\rangle)$

† E-mail address: bbakalov@fmi.uni-sofia.bg
‡ E-mail address: horozov@fmi.uni-sofia.bg
§ E-mail address: myakimov@fmi.uni-sofia.bg

0305-4470/96/175565+09$19.50

is an eigenvector of A in the (completed) bosonic Fock space. We take A to be $\zeta\partial_\zeta$ (recall that Gr is built from the space of formal Laurent series in ζ) and impose on W to be invariant under the multiplication by ζ^N (hence τ_W is a solution of the Nth reduction of KP hierarchy). These restrictions yield a compatibility condition, satisfied by W. Thus we come to the other classical object—Meijer's differential equation [6] (see (8) below) which is connected to the above modules. In the last section of the paper we give explicit formulae for the singular vectors in these modules and point out the embeddings among reducible ones.

Although we consider that these representations have their own value, we have to point out that our first motivation in their construction was the solution of the so-called bispectral problem (see [9, 28, 13] and references therein). Starting with the highest weight vectors of these modules we build broad classes of solutions of any rank to this problem (see [5] and references therein). But what we find more important is that these modules provide a natural representation-theoretic setting for many results in the bispectral problem including those of [9]. In this way we exhibit a completely new area of applications of the crucial idea of Sato: the interplay between representation theory of infinite-dimensional Lie algebras and soliton equations.

2. Preliminaries on $W_{1+\infty}$ and Sato's Grassmannian

An adequate representation-theoretic model for the Dirac sea is the infinite wedge space $F = \oplus_{m\in\mathbb{Z}} F^{(m)}$, defined as follows [17, 18, 20]. Let $V = \oplus_{j\in\mathbb{Z}}\mathbb{C}v_j$ be infinite-dimensional vector space with basis v_j. Then $F^{(m)}$ for $m \in \mathbb{Z}$ is the linear span of all semi-infinite monomials

$$v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots$$

such that $i_0 > i_1 > \cdots$ and $i_k = m - k$ for $k \gg 0$. The free fermions can be realized as wedging and contracting operators:

$$\psi_{-j+\frac{1}{2}}(v_{i_0} \wedge v_{i_1} \wedge \dots) = v_j \wedge v_{i_0} \wedge v_{i_1} \wedge \dots$$
$$\psi^*_{j-\frac{1}{2}}(v_j \wedge v_{i_0} \wedge v_{i_1} \wedge \dots) = v_{i_0} \wedge v_{i_1} \wedge \dots.$$

The Lie algebra gl_∞ of all $\mathbb{Z}\times\mathbb{Z}$ matrices, having only a finite number of non-zero entries, can be represented in F via

$$r(E_{ij}) = \psi_{-i+\frac{1}{2}}\psi^*_{j-\frac{1}{2}}$$

(where $E_{ij}v_k = \delta_{jk}v_i$). We shall need, however, a larger Lie algebra $\widetilde{gl}_\infty$ of $\mathbb{Z}\times\mathbb{Z}$ matrices with a finite number of non-zero diagonals. The above representation does not make sense for $\widetilde{gl}_\infty$. It must be regularized and leads to a representation

$$\hat{r}(E_{ij}) = {:}\psi_{-i+\frac{1}{2}}\psi^*_{j-\frac{1}{2}}{:} \tag{1}$$

of a central extension $\widehat{gl}_\infty = \widetilde{gl}_\infty \oplus \mathbb{C}c$ with central charge $c = 1$ (as usual ${:}\psi_\mu\psi^*_\nu{:} = \psi_\mu\psi^*_\nu$ for $\nu > 0$ and $= -\psi^*_\nu\psi_\mu$ for $\nu < 0$). Introduce free fermionic fields

$$\psi(z) = \sum_{j\in\mathbb{Z}} \psi_{j-\frac{1}{2}} z^{-j} \qquad \psi^*(z) = \sum_{j\in\mathbb{Z}} \psi^*_{j-\frac{1}{2}} z^{-j}$$

and $U(1)$ current

$$J(z) = {:}\psi^*(z)\psi(z){:} = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}.$$

The modes J_n generate the Heisenberg algebra and each $F^{(m)}$ is its irreducible representation with charge m and central charge $c = 1$. This gives an isomorphism, known as the boson–fermion correspondence (see e.g. [18, 20, 16])

$$\sigma : F \to B = \mathbb{C}[t_1, t_2, t_3, \ldots ; Q, Q^{-1}] \tag{2}$$

$$J_n = \frac{\partial}{\partial t_n} \qquad J_{-n} = n t_n \quad \text{for } n > 0 \qquad J_0 = Q \frac{\partial}{\partial Q}. \tag{3}$$

Introducing the states $|m\rangle = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots$ and the operator $H(t) = -\sum_{k=1}^{\infty} t_k J_k$, we have for $|\varphi\rangle \in F$

$$\sigma(|\varphi\rangle) = \sum_{m\in\mathbb{Z}} \langle m| \mathrm{e}^{H(t)} |\varphi\rangle Q^m.$$

Let

$$\phi(z) = \hat{q} + J_0 \log z + \sum_{n\neq 0} J_n \frac{z^{-n}}{-n}$$

be a scalar bosonic field with OPE $\phi(z_1)\phi(z_2) \sim \log(z_1 - z_2)$, such that

$$J(z) = \partial\phi(z) \qquad Q^m = \lim_{z\to 0} {:}\mathrm{e}^{m\phi(z)}{:}|0\rangle.$$

Then the fermionic fields $\psi(z), \psi^*(z)$ act on B via

$$\psi^*(z) = {:}\mathrm{e}^{\phi(z)}{:} \qquad \psi(z) = {:}\mathrm{e}^{-\phi(z)}{:}$$

(as usual ${:}J_n J_m{:} = J_n J_m$ if $m > n$ and $= J_m J_n$ if $m < n$, ${:}\hat{q} J_0{:} = {:}J_0 \hat{q}{:} = \hat{q} J_0$).

The Lie algebra $W_{1+\infty} = \widehat{\mathcal{D}}$ is the unique central extension of the Lie algebra $\mathcal{D}$ of complex regular differential operators on the circle [17, 19]. Denoting by c the central element of $W_{1+\infty}$ we introduce a basis

$$c, J^l_k = W(-z^{k+l}\partial^l_z) \qquad k \in \mathbb{Z} \qquad l \geqslant 0$$

and another basis

$$c, L^l_k = W(-z^k D^l_z) \qquad k \in \mathbb{Z} \qquad l \geqslant 0$$

where $D_z \equiv D = z\partial_z$. Here and further for $A \in \mathcal{D}$ we denote by $W(A)$ the corresponding element from $W_{1+\infty}$. The commutator in $W_{1+\infty}$ can be written most conveniently by the generating series [19]

$$[W(z^k \mathrm{e}^{xD}), W(z^m \mathrm{e}^{yD})](\mathrm{e}^{xm} - \mathrm{e}^{yk}) W(z^{k+m} \mathrm{e}^{(x+y)D}) + \delta_{k,-m} \frac{\mathrm{e}^{xm} - \mathrm{e}^{yk}}{1 - \mathrm{e}^{x+y}} c. \tag{4}$$

If we fix the basis $v_j = z^{-j}$ of V we can consider any operator $A \in \mathcal{D}$ as an element of $\widetilde{gl}_\infty$. This gives an embedding $\phi_0 : \mathcal{D} \hookrightarrow \widetilde{gl}_\infty$ which can be extended to an embedding [19] $\hat{\phi}_0 : W_{1+\infty} \hookrightarrow \widehat{gl}_\infty$ with $\hat{\phi}_0(c) = c$. Using (1) we obtain for $c = 1$ a free-field realization

$$W(A) = \mathrm{Res}_{z=0}\, {:}\psi(z) A \psi^*(z){:}$$

for $A \in \mathcal{D}$. Introducing the fields

$$J^l(z) = \sum_{k\in\mathbb{Z}} J^l_k z^{-k-l-1}$$

we get

$$J^l(z) = {:}(\partial^l \psi^*)(z)\psi(z){:}$$

and we have a bosonic realization

$$J^l(z) = {:}\mathrm{e}^{-\phi(z)} \frac{\partial_z^{l+1}}{l+1} \mathrm{e}^{\phi(z)}{:}. \tag{5}$$

Later we shall need a slight modification of this realization. Let $u(z) = u_0 \log z + \sum_{n\neq 0} u_n \frac{z^{-n}}{-n}$ be a constant series and replace in (5) $\phi(z)$ by $\phi(z)+u(z)$. Considering $A \in \mathcal{D}$ in the basis $v_j = \mathrm{e}^{u(z)} z^{-j}$ gives an embedding $\phi_u \colon \mathcal{D} \hookrightarrow \widetilde{gl}_\infty$ (i.e. $\phi_u(A) = \phi_0(\mathrm{e}^{-u} A \mathrm{e}^{u})$) and its extension $\hat{\phi}_u \colon W_{1+\infty} \hookrightarrow \widehat{gl}_\infty$ is

$$(\hat{r} \circ \hat{\phi}_u) J^l(z) = :\mathrm{e}^{-\phi(z)-u(z)} \frac{\partial_z^{l+1}}{l+1} \mathrm{e}^{\phi(z)+u(z)}:.$$

For example when $u(z) = s \log z$ we obtain the embedding $\hat{\phi}_s$ from [19]:

$$\hat{\phi}_{s \log z}(W(z^k \mathrm{e}^{xD})) = \sum_{j\in\mathbb{Z}} \mathrm{e}^{x(s-j)} E_{j-k,j} - \delta_{k,0} \frac{\mathrm{e}^{sx}-1}{\mathrm{e}^x - 1} c. \tag{6}$$

Now we shall briefly recall the Plücker embedding of the Sato Grassmannian in the projectivization of the infinite wedge space [18, 25]. Let

$$\tilde{V} = \left\{ \sum_{k\in\mathbb{Z}} a_k v_k \middle| a_k = 0 \text{ for } k \ll 0 \right\}$$

be the space of formal series. The Sato Grassmannian Gr consists of all subspaces $W \subset \tilde{V}$ which have an admissible basis

$$w_k = v_k + \sum_{i>k} w_{ik} v_k \qquad k = 0, -1, -2, \ldots$$

(we consider only transversal subspaces).

Denote by $\tilde{F}$ and $\tilde{B}$ the formal completions of F and B. Then to the plane $W \in Gr$ we associate a state $|W\rangle \in \tilde{F}^{(0)}$ as follows:

$$|W\rangle = w_0 \wedge w_{-1} \wedge w_{-2} \wedge \ldots.$$

A change of admissible basis results in multiplication of $|W\rangle$ by a nonzero constant. The tau-function of W is the image of $|W\rangle$ under the boson–fermion correspondence:

$$\tau_W(t) = \sigma(|W\rangle) = \langle 0|\mathrm{e}^{H(t)}|W\rangle.$$

It is a formal power series in $t = (t_1, t_2, t_3, \ldots)$.

Finally, recall that the Baker (or wave) function of $W \in Gr$ is a formal series of the form

$$\psi_W(x, z) = \mathrm{e}^{xz}\left(1 + \sum_{i=1}^{\infty} a_i(x) z^{-i}\right)$$

such that

$$w_{-k} = \partial_x^k \psi_W(x, z)|_{x=0} \qquad k = 0, 1, 2, \ldots$$

is an admissible basis of W when $v_j = z^{-j}$. It is expressed by the tau-function via

$$\psi_W(x, z) = \mathrm{e}^{xz} \frac{\tau_W(x - [z^{-1}])}{\tau_W(x)} \tag{7}$$

where $\tau_W(x) = \tau_W(x, 0, 0, \ldots)$, $[z^{-1}] = (z^{-1}, z^{-2}/2, z^{-3}/3, \ldots)$.

3. Generalized Bessel functions

Fix $\boldsymbol{r} \in \mathbb{C}^N$ and let

$$P_{\boldsymbol{r}}(D) = (D - r_1)(D - r_2) \cdots (D - r_N).$$

Consider the differential equation

$$P_{\boldsymbol{r}}(D_z)\Phi(z) = z\Phi(z). \tag{8}$$

After the substitution $z = \zeta^N$, it becomes an equation in ζ with two singular points: regular at $\zeta = 0$ and irregular of rank 1 at $\zeta = \infty$ (see e.g. [27]). For every sector S with a centre at $\zeta = \infty$ and an angle less than 2π, it has a solution with asymptotics

$$\Phi(z) \sim \Psi_{\boldsymbol{r}}(\zeta) = \zeta^{s(\boldsymbol{r})} \mathrm{e}^{N\zeta} \left(1 + \sum_{i=1}^{\infty} a_i(\boldsymbol{r})\zeta^{-i}\right) \qquad \text{for } |\zeta| \to \infty,\ \zeta \in S,\ \zeta = z^{1/N}. \tag{9}$$

The formal (divergent) series $\Psi_{\boldsymbol{r}}(\zeta)$ is uniquely determined by (8) and does not depend on the sector S. The other solutions of (8) are obtained by replacing ζ by $\mathrm{e}^{2\pi\mathrm{i}k/N}\zeta$ $(0 \leqslant k \leqslant N-1)$. We have

$$s(\boldsymbol{r}) = \sum_{i=1}^{N} r_i - \frac{N-1}{2} \qquad a_0(\boldsymbol{r}) = 1$$

$$a_1(\boldsymbol{r}) = \sum_{i<j} r_i r_j - \frac{N-1}{2N}\left(\sum r_i\right)^2 + \frac{N^2-1}{24N}$$

and all $a_i(\boldsymbol{r})$ are symmetric polynomials in $r_1, \ldots, r_N$.

Example 1. (i) When $N = 2$, $\boldsymbol{r} = (\alpha/2, -\alpha/2)$, $x = 2z^{1/2}$, equation (8) becomes the classical Bessel equation

$$x^2\partial_x^2\Phi + x\partial_x\Phi - (x^2 + \alpha^2)\Phi = 0$$

and the solution with asymptotics (9) can be taken as the Bessel function of the third kind K_α (see e.g. [6]).

(ii) For all N we can take $\Phi(z) = G_{0N}^{N0}((-1)^N z|\boldsymbol{r})$ be the Meijer's G-function (see e.g. [6, 23]). When $r_i - r_j \notin \mathbb{Z}$ for all $i \neq j$ it can be expressed in terms of generalized hypergeometric functions.

We shall need some elementary properties of $\Psi_{\boldsymbol{r}}(\zeta)$, which follow directly from (8). They correspond to classical properties of Meijer's G-function (see [6], section 5.3). Here and further we denote by $\boldsymbol{e}_i$ the vector from $\mathbb{C}^N$ with 1 on the ith place and 0's elsewhere, and $\boldsymbol{e} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{e}_i$. Then we have

$$\zeta^s\Psi_{\boldsymbol{r}}(\zeta) = \Psi_{\boldsymbol{r}+s\boldsymbol{e}}(\zeta) \tag{10}$$

$$(D_z - r_i)\Psi_{\boldsymbol{r}}(\zeta) = \Psi_{\boldsymbol{r}+\boldsymbol{e}_i}(\zeta). \tag{11}$$

4. Construction of highest weight modules

From now on we fix $s \in \mathbb{C}$, $N \in \mathbb{N}$ and consider $\boldsymbol{r} \in \mathbb{C}^N$ such that $s(\boldsymbol{r}) = s$. Recall that $z = \zeta^N$. We choose a basis in V

$$v_k = \mathrm{e}^{N\zeta}\zeta^{s-k} \qquad k \in \mathbb{Z}. \tag{12}$$

Then $\Psi_{\boldsymbol{r}}(\zeta)$ corresponds to the following element of $\tilde{V}$:

$$w_0 = \Psi_{\boldsymbol{r}}(\zeta) = v_0 + \sum_{i=1}^{\infty} a_i(\boldsymbol{r}) v_i. \tag{13}$$

We define *Bessel's plane* [10] $W_{\boldsymbol{r}} \in Gr$ to be the unique plane, containing $\Psi_{\boldsymbol{r}}(\zeta)$ and invariant under D_z, and denote by $\tau_{\boldsymbol{r}}$ its tau-function. By (11) we can construct an admissible basis $\{w_{-k}\}_{k\geqslant 0}$ of $W_{\boldsymbol{r}}$ as follows. Choose arbitrary vectors $\boldsymbol{e}^{(k)} = \sum_{i=1}^{N} k_i \boldsymbol{e}_i$ with $k_i \in \mathbb{Z}_{\geqslant 0}$, $\sum_{i=1}^{N} k_i = k$ and set

$$w_{-k} = \Psi_{\boldsymbol{r}+\boldsymbol{e}^{(k)}}(\zeta) = v_{-k} + \sum_{i>0} a_i(\boldsymbol{r} + \boldsymbol{e}^{(k)}) v_{i-k}. \tag{14}$$

The Baker function of the plane $W_{\boldsymbol{r}}$ is

$$\psi_{\boldsymbol{r}}(x, z) = \mathrm{e}^{-N\zeta} \left(\zeta\left(1 + \frac{x}{N}\right)\right)^{-s} \Psi_{\boldsymbol{r}}\left(\zeta\left(1 + \frac{x}{N}\right)\right). \tag{15}$$

By (8) it follows that $W_{\boldsymbol{r}}$ is invariant under the operators z and D_z, i.e.

$$z W_{\boldsymbol{r}} \subset W_{\boldsymbol{r}} \qquad D_z W_{\boldsymbol{r}} \subset W_{\boldsymbol{r}}. \tag{16}$$

The algebra $W_{1+\infty}$ is isomorphic to its subalgebra consisting of elements of degrees divisible by N [12, 26]. The isomorphism is given explicitly by

$$\begin{aligned} \pi_N &: W(z^k \mathrm{e}^{xD_z}) \mapsto W(\zeta^{Nk} \mathrm{e}^{\frac{x}{N} D_\zeta}) + \delta_{k,0} \left(\frac{1}{1 - \mathrm{e}^{x/N}} - \frac{N}{1 - \mathrm{e}^{x}} \right) c \\ \pi_N &: c \mapsto Nc. \end{aligned} \tag{17}$$

Combining it with (5) we obtain a bosonic representation $\hat{r}_{s,N} = \hat{r} \circ \hat{\phi}_{N\zeta + s\log\zeta} \circ \pi_N$ of $W_{1+\infty}$ with central charge $c = N$. It is easy to see that

$$\hat{r}_{s,N}(W(z^k \mathrm{e}^{xD_z})) = \hat{r}(z^k \mathrm{e}^{xD_z}) + c_k(x) \tag{18}$$

where the action of $z^k \mathrm{e}^{xD_z}$ is taken in the basis (12), $c_k(x) = 0$ for $k > 0$ and

$$c_0(x) = \frac{1}{1 - \mathrm{e}^{x/N}} - \frac{N}{1 - \mathrm{e}^{x}} + \frac{\mathrm{e}^{sx} - 1}{1 - \mathrm{e}^{x}}. \tag{19}$$

Note that when $k \neq 0$ in (18) $\hat{r}$ can be replaced by r. From now on we shall consider only this representation (s, N being fixed) and skip the symbol $\hat{r}_{s,N}$. Denote by $\mathcal{M}_{\boldsymbol{r}}$ the module generated by $\tau_{\boldsymbol{r}}$.

Theorem 2. (i) $\tau_{\boldsymbol{r}}$ is a highest weight vector with highest weight $\lambda_{\boldsymbol{r}}$, i.e.

$$L_k^l \tau_{\boldsymbol{r}} = 0 \quad \text{for } k > 0 \qquad L_0^l \tau_{\boldsymbol{r}} = \lambda_{\boldsymbol{r}}(L_0^l) \tau_{\boldsymbol{r}}. \tag{20}$$

Here $\lambda_{\boldsymbol{r}}(L_0^l)$ are certain symmetric polynomials in $r_1, \ldots, r_N$, for example $\lambda_{\boldsymbol{r}}(L_0^0) = \sum_{i=1}^{N} r_i$, $\lambda_{\boldsymbol{r}}(L_0^1) = \frac{1}{2}\sum_{i=1}^{N} r_i^2 - \frac{1}{2}\sum_{i=1}^{N} r_i$.

(ii) The module $\mathcal{M}_{\boldsymbol{r}}$ is quasifinite with characteristic polynomials

$$P_{\boldsymbol{r},k}(D) = P_{\boldsymbol{r}}(D) P_{\boldsymbol{r}}(D-1) \cdots P_{\boldsymbol{r}}(D-k+1)$$

$(k \in \mathbb{N})$, i.e. we have

$$W(z^{-k} P_{\boldsymbol{r},k}(D_z)) \tau_{\boldsymbol{r}} = 0$$

and $P_{\boldsymbol{r},k}$ are polynomials of minimal degree with this property.

Proof. By (8) and (16) $W_{\boldsymbol{r}}$ is invariant under the operators $z^k D_z^l$ for $k, l \geqslant 0$ and $z^{-k} P_{\boldsymbol{r},k}(D_z) = (z^{-1} P_{\boldsymbol{r}}(D_z))^k$ for $k \geqslant 1$. We use the result of [21] that for $A \in \widetilde{gl}_\infty$, $W \in Gr$

$$AW \subset W \qquad \text{iff } \hat{r}(A)\tau_W = \text{constant} \times \tau_W$$

and the fact that

$$W(z^k f(D_z)) = \frac{1}{k}[W(D_z), W(z^k f(D_z))]$$

for $k \neq 0$. The polynomials $\lambda_{\boldsymbol{r}}(L_0^l)$ can be computed comparing the coefficient of $|0\rangle$ in both sides of (20) and using (14) and (19).

Example 3. For $N = 2$ and $\boldsymbol{r} = (\alpha/2, -\alpha/2)$ the Virasoro modules M_α^∞ introduced in [15, 10] are reductions of the modules $\mathcal{M}_{\boldsymbol{r}}$ (obtained by putting $t_2 = t_4 = t_6 = \cdots = 0$).

To describe $\lambda_{\boldsymbol{r}}$ introduce, following Kac and Radul [19], the generating series

$$\Delta_{\lambda_{\boldsymbol{r}}}(x) = \lambda_{\boldsymbol{r}}(W(-\mathrm{e}^{xD_z})).$$

It is proved in [19] that for a quasifinite representation with first characteristic polynomial $P_{\boldsymbol{r}}(D)$ the function

$$F(x) = (\mathrm{e}^x - 1)\Delta_{\lambda_{\boldsymbol{r}}}(x) + c$$

satisfies the equation

$$P_{\boldsymbol{r}}(\partial_x)F(x) = 0.$$

When all r_i are distinct, it follows that $F(x) = \sum_{i=1}^N a_i \mathrm{e}^{r_i x}$. Because of symmetry $a_1 = \cdots = a_N$. Using the value of $\lambda_{\boldsymbol{r}}(L_0^0)$ given by theorem 2 we get $a_1 = \cdots = a_N = 1$. But $\lambda_{\boldsymbol{r}}(L_0^l)$ are polynomials in $\boldsymbol{r}$, thus we have proved:

Theorem 4. The generating series $\Delta_{\lambda_{\boldsymbol{r}}}(x)$ is given by

$$\Delta_{\lambda_{\boldsymbol{r}}}(x) = \sum_{i=1}^N \frac{\mathrm{e}^{r_i x} - 1}{\mathrm{e}^x - 1}.$$

The irreducible module with such generating series is called in [11] a primitive module. However, our modules are irreducible only when $r_i - r_j \notin \mathbb{Z}$ for $i \neq j$ [19].

We shall show that $\tau_{\boldsymbol{r}}$ is characterized among all formal power series by the constraints (20).

Proposition 5. There exists only one (up to a constant) formal power series τ in $t_1, t_2, \ldots$ satisfying

$$J_k^l \tau = 0 \qquad J_0^l \tau = c_l \tau \tag{21}$$

for $0 < k$, $0 \leqslant l \leqslant N - 1$ and some constants c_l.

Proof. (cf [1, 15]) After taking derivatives of (21) and letting $t_1 = t_2 = \cdots = 0$ one sees inductively that all derivatives of τ at $t = 0$ vanish, hence it is determined by $\tau(0)$.

5. Embeddings of the modules $\mathcal{M}_{\boldsymbol{r}}$

Theorem 6. Let $\boldsymbol{r} \in \mathbb{C}^N, r_1 - r_2 = \alpha \in \mathbb{Z}_{\geqslant 0}$. Then $\tau_{\boldsymbol{r}+\boldsymbol{e}_1-\boldsymbol{e}_2}$ is a singular vector in the module $\mathcal{M}_{\boldsymbol{r}}$ and is given by the formula

$$\tau_{\boldsymbol{r}+\boldsymbol{e}_1-\boldsymbol{e}_2} = W(A)\tau_{\boldsymbol{r}} + \text{constant} \times \tau_{\boldsymbol{r}} \tag{22}$$

where

$$A = -(\alpha+1)\frac{z^{-1}P_{\boldsymbol{r}}(D_z)}{D_z - r_2}\left(z^{-1}P_{\boldsymbol{r}}(D_z)\right)^{\alpha}. \tag{23}$$

Therefore $\mathcal{M}_{\boldsymbol{r}+\boldsymbol{e}_1-\boldsymbol{e}_2} \hookrightarrow \mathcal{M}_{\boldsymbol{r}}$.

Proof. Because

$$L_0^1|_{t_1=x,t_2=t_3=\cdots=0} = \frac{x+N}{N}\partial_x + \frac{s(s-1)}{2N} + \frac{1-N^2}{12N},$$

(20) implies

$$\tau_{\boldsymbol{r}}(x) = \left(1+\frac{x}{N}\right)^{\frac{N}{2}\sum r_i^2 + \text{constant}}$$

(the inessential constant depends on s and N but not on $\boldsymbol{r}$). For $\lambda \in \mathbb{C}$ we consider the sum $\tau_\lambda(t) = \tau_{\boldsymbol{r}}(t) + \lambda\tau_{\boldsymbol{r}+\boldsymbol{e}_1-\boldsymbol{e}_2}(t)$ (cf [10]). Then (14) implies that

$$\tau_\lambda(t) = \sigma((\Psi_{\boldsymbol{r}} + \lambda\Psi_{\boldsymbol{r}+\boldsymbol{e}_1-\boldsymbol{e}_2}) \wedge \Psi_{\boldsymbol{r}+\boldsymbol{e}_1} \wedge \Psi_{\boldsymbol{r}+\boldsymbol{e}_1+\boldsymbol{e}^{(1)}} \wedge \ldots)$$

is a tau-function. By (7) its Baker function $\psi_\lambda(x,\zeta)$ is equal to

$$\frac{\tau_{\boldsymbol{r}}(x)\psi_{\boldsymbol{r}}(x,\zeta) + \lambda\tau_{\boldsymbol{r}+\boldsymbol{e}_1-\boldsymbol{e}_2}(x)\psi_{\boldsymbol{r}+\boldsymbol{e}_1-\boldsymbol{e}_2}(x,\zeta)}{\tau_{\boldsymbol{r}}(x) + \lambda\tau_{\boldsymbol{r}+\boldsymbol{e}_1-\boldsymbol{e}_2}(x)} = \frac{\psi_{\boldsymbol{r}}(x,\zeta) + \lambda(1+\frac{x}{N})^{N(\alpha+1)}\psi_{\boldsymbol{r}+\boldsymbol{e}_1-\boldsymbol{e}_2}(x,\zeta)}{1+\lambda(1+\frac{x}{N})^{N(\alpha+1)}}.$$

Using (15), (10), (11) and (8) it is easy to see that

$$\left(1+\frac{x}{N}\right)^{N(\alpha+1)}\psi_{\boldsymbol{r}+\boldsymbol{e}_1-\boldsymbol{e}_2}(x,\zeta) = \mathrm{e}^{-N\zeta}\zeta^{-s}A(z,D_z)\zeta^s\mathrm{e}^{N\zeta}\psi_{\boldsymbol{r}}(x,\zeta).$$

Therefore in the basis (12)

$$\tau_\lambda(t) = \sigma((1+\lambda A)\Psi_{\boldsymbol{r}} \wedge (1+\lambda A)\Psi_{\boldsymbol{r}+\boldsymbol{e}^{(1)}} \wedge (1+\lambda A)\Psi_{\boldsymbol{r}+\boldsymbol{e}^{(2)}} \wedge \ldots)$$

and comparing the coefficient of λ gives

$$\tau_{\boldsymbol{r}+\boldsymbol{e}_1-\boldsymbol{e}_2}(t) = r(A)\tau_{\boldsymbol{r}}(t).$$

The formula (22) now follows from (18).

Theorem 6 gives in practice all possible embeddings among the modules $\mathcal{M}_{\boldsymbol{r}}$.

Acknowledgments

We thank Professor I T Todorov for useful discussions and for his interest in this work. We also thank the referee for pointing out the paper [4] where the case $c = 1$ has been studied in details. This paper is partially supported by grant MM-402/94 of the Bulgarian Ministry of Education, Science and Technologies.

References

[1] Adler M and van Moerbeke P 1992 *Commun. Math. Phys.* **147** 25

[2] Adler M and van Moerbeke P 1994 *Comment. Pure Appl. Math.* **47** 5

[3] Awata H, Fukuma M, Matsuo Y and Odake S 1994 Representation theory of the $W_{1+\infty}$ algebra *Kyoto Preprint* hep-th/9408158

[4] Awata H, Fukuma M, Odake S and Quano Y-H 1994 *Lett. Math. Phys.* **31** 289

[5] Bakalov B, Horozov E and Yakimov M 1996 Highest weight modules of $W_{1+\infty}$, Darboux transformations and the bispectral problem *Proc. Conf. Geom. and Math. Phys. (Zlatograd, 1995)* q-alg/9601017

[6] Bateman H and Erdélyi A 1953 *Higher Transcendental Functions* (New York: McGraw-Hill)

[7] Date E, Jimbo M, Kashiwara M and Miwa T 1983 Transformation groups for soliton equations *Proc. RIMS Symp. Nonlinear Integrable Systems (Kyoto, 1981)* (Singapore: World Scientific)

[8] Dickey L A 1991 *Soliton Equations and Hamiltonian Systems* (Adv. Ser. Math. Phys. **12**) (Singapore: World Scientific)

[9] Duistermaat J J and Grünbaum F A 1986 *Commun. Math. Phys.* **103** 177

[10] Fastré J 1993 *Doctoral Dissertation* University of Louvain

[11] Frenkel E, Kac V, Radul A and Wang W 1995 *Commun. Math. Phys.* **170** 337, hep-th/9405121

[12] Fukuma M, Kawai H and Nakayama R 1992 *Commun. Math. Phys.* **143** 371

[13] Grünbaum F A 1994 *Comment. Pure Appl. Math.* **47** 307

[14] Haine L and Horozov E 1993 *Bull. Sci. Math.* **177** 485

[15] Haine L and Horozov E 1995 *Abelian Varieties* ed Barth *et al* (Berlin: de Gruyter)

[16] Kac V G 1990 *Infinite-dimensional Lie Algebras* 3rd edn (Cambridge: Cambridge University Press)

[17] Kac V G and Peterson D H 1981 *Proc. Nat. Acad. Sci., USA* **78** 3308

[18] Kac V G and Peterson D H 1986 *Lectures on the Infinite Wedge Representation and the MKP Hierarchy* (Sem. Math. Sup. **102**) (Montreal: University of Montreal Press)

[19] Kac V G and Radul A 1993 *Commun. Math. Phys.* **157** 429, hep-th/9308153

[20] Kac V G and Raina A 1987 *Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras* (Adv. Ser. Math. Phys. **2**) (Singapore: World Scientific)

[21] Kac V G and Schwarz A 1991 *Phys. Lett.* **257B** 329

[22] Kontsevich M 1992 *Commun. Math. Phys.* **147** 1

[23] Luke Y 1975 *Mathematical Functions and Their Approximations* (New York: Academic)

[24] Sato M 1981 *RIMS Kokyuroku* **439** 30

[25] Segal G and Wilson G 1985 *Publ. Math. IHES* **61** 5

[26] van Moerbeke P 1994 *Integrable Foundations of String Theory (CIMPA, Summer School at Sophia, Antipolis, 1991)* (Singapore: World Scientific)

[27] Wasow W 1965 *Asymptotic Expansions for Ordinary Differential Equations* (New York: Interscience)

[28] Wilson G 1993 *J. Reine Angew. Math.* **442** 177

[29] Witten E 1991 *Surv. Diff. Geom.* **1** 243